Noncommutative Geometric Gauge Theory from Superconnections

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Abstract

Noncommutative geometric gauge theory is reconstructed based on the superconnection concept. The bosonic action of the Connes-Lott model including the symmetry breaking Higgs sector is obtained by using a new generalized derivative, which consists of the usual 1-form exterior derivative plus an extra element called *the matrix derivative*, for the curvatures. We first derive the matrix derivative based on superconnections and then show how the matrix derivative can give rise to spontaneous symmetry breaking. We comment on the correspondence between the generalized derivative and the generalized Dirac operator of the Connes-Lott model.

I. INTRODUCTION

Several years ago, Connes [1] proposed to use noncommutative geometry for particle physics models. Then he and Lott [2] showed that one could obtain the standard model from a noncommutative geometric gauge theory. In this noncommutative framework, the Dirac K-cycle plays an important role. For physicists, however, this formalism demands quite a bit of mathematics. Here, we present a formalism, which we hope is relatively easier for physicists to understand, based on the superconnection concept [3,4]. Coquereaux and other people [5–8] prosposed models for a noncommutative geometric gauge theory in which they also used a matrix commutator operator similar to the one that we use in this paper. Especially, the term *matrix derivative* was coined in one of these works [6]. However, these works were based on different concepts, as we shall mention later.

In this paper, we first review the Connes-Lott approach to noncommutative geometric gauge theory for self-containedness. Then we present the superconnection approach: After a brief introduction to superconnections, we derive the matrix derivative by generalizing the exterior derivative, write the Yang-Mills action with curvatures construructed from the generalized derivative including the matrix derivative, and show how *the matrix derivative* can give rise to symmetry breaking. Finally, we comment on the correspondence between our generalized derivative and the generalized Dirac operator of the Connes-Lott approach and compare our approach with others and then make conclusions.

II. CONNES-LOTT APPROACH

In the Connes-Lott approach, the Dirac K-cycle on a * (involution) algebra acting on a Hilbert space plays an important role. Both spacetime and internal space are described by the involution algebra, and the newly introduced generalized Dirac operator is crucial for symmetry breaking, which is related to the fermionic mass matrix.

A K-cycle on a * algebra $\mathcal{A}$ is given by

• a faithful representation ρ of $\mathcal{A}$ by bounded operators on a Hilbert space $\mathcal{H}$,

• a self-adjoint (generalized Dirac) operator $\mathcal{D}$ on $\mathcal{H}$ such that $[\mathcal{D}, a]$ is a bounded operator for all $a \in \mathcal{A}$, and $(1+\mathcal{D}^2)^{-1}$ is a compact operator on $\mathcal{H}$.

In the Connes-Lott approach, an even K-cycle is used, which has, in addition,

• a self-adjoint $\mathbf{Z}_2$ grading (chirality) operator Γ on $\mathcal{H}$ such that $\Gamma^2 = 1$, $\Gamma\mathcal{D} + \mathcal{D}\Gamma = 0$, and $\Gamma a = a\Gamma$ for all $a \in \mathcal{A}$. This even K-cycle is called *the Dirac K-cycle.*

Another important ingredient of the Connes-Lott approach is the π representation of the universal differential envelop of $\mathcal{A}$, $\Omega^*(\mathcal{A})$, where $\Omega^*(\mathcal{A}) = \oplus\Omega^k(\mathcal{A})$ such that $\Omega^0(\mathcal{A}) = \mathcal{A}$ and $\Omega^k(\mathcal{A}) = \{a_0\delta a_1 \cdots \delta a_k;\ a_0, a_1, \cdots, a_k \in \mathcal{A}\}$, the space of universal k-forms. The differential δ satisfies $\delta^2 = 0$, $\delta(a_0\delta a_1 \cdots \delta a_k) = \delta a_0 \delta a_1 \cdots \delta a_k \in \Omega^{k+1}(\mathcal{A})$, and the involution $*$ is given by $(a_0\delta a_1 \cdots \delta a_k)^* = \delta a_k^* \cdots \delta a_1^* a_0^*$. Now, π is a map from $\Omega^*(\mathcal{A})$ to $\mathcal{B}(\mathcal{H})$, the space of bounded operators on $\mathcal{H}$, given by

$$\pi(a_0\delta a_1 \cdots \delta a_k) = \rho(a_0)[\mathcal{D}, \rho(a_1)] \cdots [\mathcal{D}, \rho(a_k)]. \tag{2.1}$$

Note that, in order to respect the nilpotency of δ, $\mathcal{D}$ should satisfy $[\mathcal{D}, [\mathcal{D},\ \cdot\ ]] = 0$.

Finally, the tensor product of two noncommutative spaces with Dirac K-cycles, $(\mathcal{A}_1, \mathcal{H}_1, \mathcal{D}_1, \Gamma_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, \mathcal{D}_2, \Gamma_2)$, is defined as

$$\begin{aligned} &\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2, \qquad \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2, \\ &\mathcal{D} = \mathcal{D}_1 \otimes 1 + \Gamma_1 \otimes \mathcal{D}_2 \quad (^{\text{or}} = \mathcal{D}_1 \otimes \Gamma_2 + 1 \otimes \mathcal{D}_2), \\ &\Gamma = \Gamma_1 \otimes \Gamma_2, \qquad \Omega^*(\mathcal{A}) = \Omega^*(\mathcal{A}_1) \otimes \Omega^*(\mathcal{A}_2). \end{aligned} \tag{2.2}$$

Here, the definition of $\mathcal{D}$ in the product space depends on which of the initial spaces corresponds to spacetime. For instance, if space 1 corresponds to spacetime, then we use the first definition; if space 2 does, then we use the second one. Now, we get into the models of the Connes-Lott approach.

A. Two-point space:

Take $\mathcal{A} = \mathbf{C} \oplus \mathbf{C}$, $\mathcal{H} = \mathbf{C}^N \oplus \mathbf{C}^N$, and $\mathcal{D} = \begin{pmatrix} 0 & M^\dagger \\ M & 0 \end{pmatrix}$, $\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ where M is an $N \times N$ matrix. If $a = (\lambda, \lambda') \in \mathcal{A}$ then $\pi(\delta a) = [\mathcal{D}, \rho(a)] = (\lambda - \lambda') \begin{pmatrix} 0 & -M^\dagger \\ M & 0 \end{pmatrix}$. The connection is given by $\mathcal{J} = a_0 \delta a_1$ with $a_0 = (u, u')$, $a_1 = (v, v') \in \mathcal{A}$; thus, $\pi(\mathcal{J}) = \pi(a_0 \delta a_1) = \rho(a_0)[\mathcal{D}, \rho(a_1)] = (v - v') \begin{pmatrix} 0 & -uM^\dagger \\ u'M & 0 \end{pmatrix}$. From $\mathcal{J} = a_0 \delta a_1 = (u, u') \cdot (v' - v, v - v') = (u(v' - v), u'(v - v'))$ and denoting it by $\mathcal{J} \equiv (\phi^*, \phi)$, we can write $\pi(\mathcal{J}) = \begin{pmatrix} 0 & \phi^* M^\dagger \\ \phi M & 0 \end{pmatrix}$. The π representation of the curvature θ is given by $\pi(\theta) = \pi(\delta\mathcal{J} + \mathcal{J}^2) = (|\phi+1|^2 - 1) \begin{pmatrix} M^\dagger M & 0 \\ 0 & MM^\dagger \end{pmatrix}$. Now the Yang-Mills action is given by $I_\bigtriangledown = \mathrm{Tr}_\omega((\pi(\theta))^2 \mathcal{D}_\bigtriangledown^{-n})$ where Tr_ω is the Dixmier trace, n is the dimension of the manifold, and $\mathcal{D}_\bigtriangledown$ is the "covariant derivative" given by $\mathcal{D}_\bigtriangledown = \mathcal{D} + \pi(\mathcal{J})$. Since n is zero and the Dixmier trace becomes the usual trace in the present case, the Yang-Mills action is given by $I_\bigtriangledown = 2(|\phi + 1|^2 - 1)^2 \mathrm{Tr}(M^\dagger M)^2$. This is just the Higgs potential with minima at $\phi = 0, -2$; this type of potential indicates explicitly broken symmetry.

B. Spinmanifold:

Take $\mathcal{A} = C^\infty(Z) \otimes \mathbf{C}$ where Z is a 4-dimensional spinmanifold, and let $\mathcal{H} = L^2(S)$ where S is the vector bundle of spinors on Z. Then, the Dirac and the chirality operators become the usual ones, $\mathcal{D} = \gamma^\mu \partial_\mu$ and $\Gamma = \gamma_5$. The connection is an ordinary differential 1-form on Z, $\pi(\mathcal{J}) = A$. Thus, the curvature is given as the usual $\pi(\theta) = F = dA + A^2$, and the Yang-Mills action is $I_\bigtriangledown = \int_Z \mathrm{Tr}(F * F)$.

C. Product space:

Consider now the tensor product of the above two spaces. Following the given tensor product rule of Eq. (2.2), we get

$$\begin{aligned}
&\mathcal{A} = C^\infty(Z) \otimes (\mathbf{C} \oplus \mathbf{C}), \;\; \mathcal{H} = L^2(S) \otimes (\mathbf{C}^N \oplus \mathbf{C}^N), \\
&\mathcal{D} = \gamma^\mu \partial_\mu \otimes I_2 \otimes I_N + \gamma_5 \otimes \begin{pmatrix} 0 & M^\dagger \\ M & 0 \end{pmatrix}, \\
&\Gamma = \gamma_5 \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes I_N.
\end{aligned} \tag{2.3}$$

The connection is given by $\pi(\mathcal{J}) = \begin{pmatrix} A & \phi^* \gamma_5 M^\dagger \\ \phi \gamma_5 M & A' \end{pmatrix}$. The curvature is thus given by

$$\begin{aligned}
\pi(\theta) &= \pi(\delta \mathcal{J} + \mathcal{J}^2) \\
&= \begin{pmatrix} dA + AA + (\phi + \phi^* + \phi^* \phi) M^\dagger M & (d\phi^* + A\phi^* - \phi^* A' + A - A')\gamma_5 M^\dagger \\ (d\phi - \phi A + A'\phi - A + A')\gamma_5 M & dA' + A'A' + (\phi + \phi^* + \phi\phi^*) M M^\dagger \end{pmatrix}
\end{aligned}$$

where $d = \gamma^\mu \partial_\mu$ and $A = \gamma^\mu A_\mu, \;\; A' = \gamma^\mu A'_\mu$ are a 1-form exterior derivative and gauge fields, respectively. The Yang-Mills action now mainly consists of three parts, the usual Yang-Mills action, the kinetic part of scalar field, and the Higgs potential:

$$I_\bigtriangledown \sim \int_Z \mathrm{Tr}(\alpha_1(|F_A|^2 + |F_{A'}|^2) + \alpha_2 |D\phi|^2 + \alpha_3(|\phi + 1|^2 - 1)^2 + \cdots) \tag{2.4}$$

where $D\phi = d\phi + A'\phi - \phi A$ and "$\cdots$" denotes both mass terms for $A, \;\; A'$ and interaction terms.

III. SUPERCONNECTION APPROACH

The superconnection was first introduced in mathematics by Quillen in 1985 [3]. However, in physics this concept was used earlier in 1982 by Thierry-Mieg and Ne'eman without giving it a name [9] under the notion of a generalized connection *a la* Cartan [10]. Then in

1990, Ne'eman and Sternberg [4] used superconnections for a Higgs mechanism in a manner for physicists to understand this concept much easier than that of Quillen's. We now follow the Ne'eman-Sternberg presentation of superconnections in this paper.

Let $V = V^+ \oplus V^-$ be a super (or Z_2-graded) complex vector space; then, the algebra of endomorphisms of V is a superalgebra with even or odd endomorphisms. Let $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ be a super (or Z_2-graded) vector bundle over a manifold M, and $\Omega(M) = \oplus\Omega^k(M)$ be the algebra of smooth differential forms with complex coefficients. Then, let $\Omega(M,\mathcal{E})$ be the space of $\mathcal{E}$-valued differential foms on M. This space has a $Z \times Z_2$ grading; however, we are mainly concerned with its total Z_2 grading. Hence, $\Omega(M,\mathcal{E})$ can be regarded as a supermodule over $\Omega(M)$. The total Z_2 grading can be denoted by $\Omega(M,\mathcal{E}) = \Omega^+(M,\mathcal{E}) + \Omega^-(M,\mathcal{E})$ which is defined by

$$\Omega^\pm(M,\mathcal{E}) = \sum_k \Omega^{2k}(M,\mathcal{E}^\pm) \oplus \sum_k \Omega^{2k+1}(M,\mathcal{E}^\mp)$$

where $2k$ $(2k+1)$ indicates an even (odd) exterior form degree. Now we consider the tensor product of $\Omega(M)$ and $\mathcal{A} \equiv \mathrm{End}(V)$, which belongs to $\mathrm{End}(\Omega(M,\mathcal{E}))$. We decompose $\mathcal{A}$ as $\mathcal{A} = \mathcal{A}^+ \oplus \mathcal{A}^-$ such that $\mathcal{A}^+$ consists of all "matrices" of the form

$$\begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix}, \quad R \in \mathrm{End}(V^+), \quad S \in \mathrm{End}(V^-),$$

while $\mathcal{A}^-$ consists of all "matrices" of the form

$$\begin{pmatrix} 0 & K \\ L & 0 \end{pmatrix}, \quad K \in \mathrm{Hom}(V^-,\ V^+), \quad L \in \mathrm{Hom}(V^+,\ V^-).$$

If we choose bases of V^+ and V^-, then we can think of $R,\ S,\ K,$ and L as actual matrices. We can, thus, think of elements of $\Omega(M) \otimes \mathcal{A}$ as matrices whose entries are differential forms. For instance, both $\begin{pmatrix} \omega_0 & 0 \\ 0 & \omega_1 \end{pmatrix}$ and $\begin{pmatrix} 0 & L_{01} \\ L_{10} & 0 \end{pmatrix}$ are odd elements of $\Omega(M) \otimes \mathcal{A}$ if $\omega_0,\ \omega_1$ are matrices of odd degree differential forms and $L_{01},\ L_{10}$ are matrices of even degree differential forms.

A superconnection $\bigtriangledown$ on $\mathcal{E}$ is an odd element of $\mathrm{End}(\Omega(M,\mathcal{E}))$, that is,

$$\bigtriangledown \; : \; \Omega^{\pm}(M,\mathcal{E}) \longrightarrow \Omega^{\mp}(M,\mathcal{E}), \tag{3.1}$$

and satisfies the derivation property

$$\bigtriangledown (v\alpha) = (dv)\alpha + (-1)^{|v|} v \bigtriangledown \alpha, \quad v \in \Omega(M), \;\; \alpha \in \Omega(M,\mathcal{E}) \tag{3.2}$$

where d is a 1-form exterior derivative operator which is odd, and $|v|$ is the exterior degree of v. In terms of the local trivialization of $\mathcal{E}$, say $\mathcal{E} = M \times V$ (locally), the most general superconnection can be written locally as

$$\bigtriangledown = d + \omega, \quad \omega \in (\Omega(M) \otimes \mathcal{A})^{-} = \Omega^{+}(M) \otimes \mathcal{A}^{-} \oplus \Omega^{-}(M) \otimes \mathcal{A}^{+}. \tag{3.3}$$

In "matrix" language, d is given by $\mathbf{d} = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}$ with d inside the matrix denoting the usual 1-form exterior derivative operator times a unit matrix, and ω is given by $\omega = \begin{pmatrix} \omega_0 & L_{01} \\ L_{10} & \omega_1 \end{pmatrix}$ where $\omega_0, \; \omega_1$ are matrices of odd degree differential forms and $L_{01}, \; L_{10}$ are matrices of even degree differential forms. Here, the multiplication rule is given by

$$(u \otimes a) \cdot (v \otimes b) = (-1)^{|a||v|} (uv) \otimes (ab), \quad u, v \in \Omega(M), \;\; a, b \in \mathcal{A}. \tag{3.4}$$

In what follows, we will use this matrix language.

We now derive the way in which *the matrix derivative* enters into play in this superconnection formulation. First, we note that the 1-form exterior derivative $\mathbf{d}$ is nilpotent and satisfies the derivation rule

$$\mathbf{d}^2 = 0, \quad \mathbf{d}(\alpha\beta) = (\mathbf{d}\alpha)\beta + (-1)^{|\alpha|}\alpha(\mathbf{d}\beta), \quad \alpha, \beta \in \Omega(M,\mathcal{E}) \tag{3.5}$$

where $|\alpha|$ is the total Z_2 grading of α. Next, we want to generalize this operator acting on $\Omega(M,\mathcal{E})$, say $\mathbf{d}_t$, keeping the above two properties such that

$$\mathbf{d}_t^2 = 0, \quad \mathbf{d}_t(\alpha\beta) = (\mathbf{d}_t\alpha)\beta + (-1)^{|\alpha|}\alpha(\mathbf{d}_t\beta), \quad \alpha, \beta \in \Omega(M,\mathcal{E}). \tag{3.6}$$

To make things easy, we look for an object that can be added to $\mathbf{d}$ as a component of $\mathbf{d}_t$, and write this additional component as $\mathbf{d}_M$: $\mathbf{d}_t = \mathbf{d} + \mathbf{d}_M$. Thus, the two conditions that $\mathbf{d}_t$ should satisfy now become

$$\mathbf{d}_M^2 = 0, \qquad \mathbf{d}\mathbf{d}_M + \mathbf{d}_M\mathbf{d} = 0, \tag{3.7}$$

$$\mathbf{d}_M(\alpha\beta) = (\mathbf{d}_M\alpha)\beta + (-1)^{|\alpha|}\alpha(\mathbf{d}_M\beta), \quad \alpha, \beta \in \Omega(M, \mathcal{E}).$$

Since $\mathbf{d}_M$ should behave as a part of the superconnection operator [11] in a sense, we write it as a (graded) commutator operator

$$\mathbf{d}_M = [\eta,\ \cdot\ ], \quad \eta \in \Omega(M, \mathcal{E}) \tag{3.8}$$

and fix η such that $\mathbf{d}_M$ satisfies the conditions in Eq. (3.7). The derivation rule and the condition $\mathbf{d}\mathbf{d}_M + \mathbf{d}_M\mathbf{d} = 0$ require that η be odd and $\mathbf{d}\eta = 0$; i.e. $|\eta| = 1$ and the elements of η should be closed forms. The nilpotency condition $\mathbf{d}_M^2 = 0$ is satisfied if η^2 commutes with any element in $\Omega(M, \mathcal{E})$. There are two simple choices satisfying this condition:

- $\eta = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ where $u,\ v$ are odd degree closed forms with their coefficient matrices satisfying $u^2 = v^2 \propto 1$, or
- $\eta = \begin{pmatrix} 0 & m \\ n & 0 \end{pmatrix}$ where $m,\ n$ are even degree closed forms with their coefficient matrices satisfying $mn = nm \propto 1$.

In this paper, we choose the second one with 0-form $m,\ n$. We express this choice by $\mathbf{d}_M = [\eta,\ \cdot\ ]$ with $\eta = \begin{pmatrix} 0 & \zeta \\ \overline{\zeta} & 0 \end{pmatrix}$ where $\zeta,\ \overline{\zeta}$ are 0-form constant matrices satisfying $\zeta\overline{\zeta} = \overline{\zeta}\zeta \propto 1$. We call this operator $\mathbf{d}_M$ *the matrix derivative* following the terminology used in Ref. 6.

With the use of this generalized derivative, the curvature is now given by

$$\mathcal{F}_t = (\mathbf{d}_t + \omega)^2 = \mathbf{d}_t\omega + \omega^2. \tag{3.9}$$

In this formulation, we write the Yang-Mills action as

$$I_t = \int_M \mathrm{Tr}(\mathcal{F}_t^\star \mathcal{F}_t) \tag{3.10}$$

where $\star$ denotes taking dual for each entry of $\mathcal{F}_t$ in addition to taking the Hermitian conjugate. In the next section, it will be shown that this action will provide the Yang-Mills-Higgs action with spontaneously broken symmetry that we obtained in the previous section through the Connes-Lott approach.

IV. SYMMETRY BREAKING AND COMPARISON BETWEEN THE TWO APPROACHES

In order to compare the action I_t obtained in the previous section with the one given in Section II, we assign 0-form scalar fields in the odd part and 1-form gauge fields in the even part of ω of the superconnection:

$$\omega = \begin{pmatrix} A & \phi^* \\ \phi & A' \end{pmatrix}. \tag{4.1}$$

Now, we choose $\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for simplicity. Then, the curvature is given by

$$\mathcal{F}_t = \begin{pmatrix} F_A + (\phi^*\phi + \phi + \phi^*) & D\phi^* + A - A' \\ D\phi + A' - A & F_{A'} + (\phi\phi^* + \phi + \phi^*) \end{pmatrix} \tag{4.2}$$

where $F_A = dA + A^2$, $F_{A'} = dA' + A'^2$, $D\phi = d\phi + A'\phi - \phi A$, and $D\phi^* = d\phi^* + A\phi^* - \phi^* A'$. Thus, the Yang-Mills action in the superconnection approach is given by

$$\begin{aligned} I_t &= \int_M \mathrm{Tr}(\mathcal{F}_t^\star \mathcal{F}_t) \\ &\sim \int_M \mathrm{Tr}\left[|F_A|^2 + |F_{A'}|^2 + 2(|D\phi|^2 + (|\phi+1|^2 - 1)^2) + \cdots\right] \end{aligned} \tag{4.3}$$

where "$\cdots$" includes the mass terms for A, A' and interaction terms. Now, the Higgs potential clearly shows that symmetry is broken explicitely [5,12,13]. This action is the same as the one we obtained in the product space case of the Connes-Lott approach in Section II.

Now, we would like to compare the generalized derivative in the superconnection approach with the generalized Dirac operator in the Connes-Lott approach. The generalized Dirac operator in the product space case in Section II was given by

$$\mathcal{D} = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} + \begin{pmatrix} 0 & \gamma_5 M^\dagger \\ \gamma_5 M & 0 \end{pmatrix}, \tag{4.4}$$

and the differential δ translates into a commutator operator, $[\mathcal{D},\ \cdot\ ]$, in the π representation. In the superconnection approach, the 1-form exterior derivative can be written as $\mathbf{d} = [\mathbf{d},\ \cdot\ ]$ since

$$[\mathbf{d}, \alpha] = \mathbf{d}\alpha - \alpha\mathbf{d} = (\mathbf{d}\alpha), \quad \alpha \in \Omega(M, \mathcal{E}). \tag{4.5}$$

Therefore, the generalized derivative $\mathbf{d}_t = \mathbf{d} + \mathbf{d}_M$ can be expressed as

$$\mathbf{d}_t = [\mathbf{d},\ \cdot\ ] + \mathbf{d}_M. \tag{4.6}$$

Remember that $\mathbf{d} = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}$ and $\mathbf{d}_M = [\eta,\ \cdot\ ]$ where $\eta = \begin{pmatrix} 0 & \zeta \\ \overline{\zeta} & 0 \end{pmatrix}$. Thus, we can see that the generalized Dirac operator in the two-point space case plays the role of the matrix derivative in the superconnection approach. Also, the name *fermionic mass matrix* for M in the two-point space case will be clear if we look into the fermionic action

$$I_\Psi = \int_Z \overline{\Psi} \mathcal{D}_\bigtriangledown \Psi \tag{4.7}$$

where $\mathcal{D}_\bigtriangledown = \mathcal{D} + \pi(\mathcal{J})$. If we write $\Psi = (\psi_L, \psi_R)$, then the generalized Dirac operator induces the mass terms $\overline{\psi}_L \gamma_5 M^\dagger \psi_R + \overline{\psi}_R \gamma_5 M \psi_L$. In the superconnection approach, the fermionic action is given by

$$I_\Psi = \int_M \overline{\Psi}(\mathbf{d}_t + \omega)\Psi, \quad \Psi \in V \otimes S \tag{4.8}$$

where S is a spinor bundle. Since V is a Z_2-graded (super) vector space, we can write $\Psi = (\psi_+, \psi_-)$. Then the matrix derivative term induces the mass terms, $\overline{\psi}_+ \zeta \psi_- + \overline{\psi}_- \overline{\zeta} \psi_+$, in this case also.

V. COMMENTS AND CONCLUSIONS

As we have seen, the role of the matrix derivative in the symmetry-breaking mechanism is very similar to that of the generalized Dirac operator. Without introducing the generalized Dirac operator, whose matrix component is called the fermionic mass matrix, it was not

possible to include the scalar field as a part of the connection in the Connes-Lott approach. Also, as its component name, *fermionic mass matrix*, suggests, the generalized Dirac operator is essential for spontaneous symmetry breaking in that approach. Similarly, although the scalar field is a part of the superconnection from the beginning in the superconnection approach, there would be no "automatic" symmetry breaking without the matrix derivative [14,15].

Our approach looks similar to some approaches presented earlier [5,6] in the sense that there too the "matrix derivative" was used in conjunction with Connes-Lott's two-point space idea. Especially, in one of them, Ref. 6, the term "matrix derivative" was coined. However, in those works, this matrix commutator operator was simply defined to conform with the generalized Dirac operator and to induce the Connes-Lott action, as their main architect Coquereaux confirms [16]. Thus, the scalar field was regarded as a 1-form in Coquereaux *et al.*'s approach [16], as it should be in the Connes-Lott approach, rather than as the 0-form that it should be in the superconnection approach.

We also note that the notion of a Z_2-graded (super) vector space which is essential to the superconnection construction was crucial for the correspondence between this approach and the Connes-Lott approach. This is because in the Connes-Lott approach the two-point space structure is very basic in obtaining the standard model since it induces the left and the right movers even if one deals with a more complicated structure [17]. In the superconnection approach this requirement is provided by the Z_2-graded vector space [18].

In conclusion, here we have shown the way in which the generalization of the exterior derivative in the superconnection framework can provide the so-called matrix derivative, and we have produced the Yang-Mills-Higgs action with built-in symmetry breakdown mechanism, which was obtained from the Connes-Lott model through the generalized Dirac operator.

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REFERENCES

[1] A. Connes, *The Interface of Mathematics and Particle Physics,* edited by D. Quillen, G. Segal, and S. Tsou (Oxford Univ. Press, Oxford, 1990), p. 9.

[2] A. Connes and J. Lott, Nucl. Phys. B (Proc. Suppl.) **18B**, 29 (1990); *New Symmetry Principles in Quantum Field Theory,* edited by J. Frörich and G. 't Hooft (Plenum Press, New York, 1992), p. 53.

[3] D. Quillen, Topology **24**, 89 (1985).

[4] Y. Ne'eman and S. Sternberg, Proc. Natl. Acad. Sci. USA **87**, 7875 (1990).

[5] R. Coquereaux, G. Esposito-Farèse, and G. Vaillant, Nucl. Phys. **B353**, 689 (1991); R. Coquereaux, G. Esposito-Farèse, and F. Scheck, Int. J. Mod. Phys. **A7**, 6555 (1992); R. Coquereaux, R. Häußling, and F. Scheck, Int. J. Mod. Phys. **A10**, 89 (1995).

[6] R. Häußling, N. A. Papadopoulos, and F. Scheck, Phys. Lett. **B260**, 125 (1991).

[7] N. A. Papadopoulos, J. Plass, and F. Scheck, Phys. Lett. **B324**, 380 (1994).

[8] F. Hussain and G. Thompson, Phys. Lett. **B260**, 359 (1991); *ibid.* **B265**, 307 (1991).

[9] J. Thierry-Mieg and Y. Ne'eman, Proc. Natl. Acad. Sci. USA **79**, 7068 (1982).

[10] E. Cartan, C. R. Acad. Sci. Paris **182**, 956 (1926).

[11] N. Berlin, E. Getzler, and M. Vergne, *Heat Kernels and Dirac Operators,* (Springer, Berlin, 1992), Chap. 1.

[12] C. Y. Lee, D. S. Hwang, and Y. Ne'eman, *hep-th 9512215*, J. Math. Phys. **37**, 3725 (1996).

[13] D. S. Hwang and T. Lee, Int. J. Mod. Phys. **A9**, 5531 (1994).

[14] C. Y. Lee and Y. Ne'eman, Phys. Lett. **B264**, 389 (1991); *ibid.* **B269**, 477 (1991).

[15] D. S. Hwang, C. Y. Lee, and Y. Ne'eman, Int. J. Mod. Phys. **A11**, 3509 (1996).

[16] G. Cammarata and R. Coquereaux, *hep-th 9505192*.

[17] B. Iochum and T. Schücker, *hep-th 9501142*.

[18] S. Cho and K. S. Park, J. Korean Phys. Soc. **27**, 255 (1994); J. H. Lee, C. K. Jue, and K. H. Lee, J. Korean Phys. Soc. **27**, 21 (1994); D. S. Soh, K. H. Cho, and S. P. Kim, J. Korean. Phys. Soc. **28**, 777 (1995).